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Triangulated Categories

This note is prepared for an introduction to triangulated category based on Neeman's book.

History : Triangulated categories were defined independently and around the same time by Puppe and Jean-Louis Verdier (1963). Verdier's original work was in his PhD thesis based on the ideas of Grothendieck.

Additive Categories

Definition: A category $\mathcal{C}$ is *preadditive* if:

1) For every pair of objects of $\mathcal{C}$ like A, B Hom(A, B) has the structure of an abelian group
2) composition of morphisms is bilinear ,i.e.
   $\forall f, f' \in Hom(A,B)\ ,\ \forall g, g' \in Hom(B,C)\ :$

$$g \circ (f + f') = g \circ f + g \circ f',$$
$$(g + g') \circ f = g \circ f + g' \circ f.$$

Definition:

A diagram

$$X \underset{i_1}{\overset{p_1}{\leftrightarrows}} Z \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$$

is a *biproduct* of X and Y if
$p_1 \circ i_1 = 1_X,\ p_2 \circ i_2 = 1_Y\ \ and\ i_1 \circ p_1 + i_2 \circ p_2 = 1_Z.$

Definition: A category $\mathcal{C}$ is *additive* if

1) is preadditive;
2) has a zero object(is pointed) ;
3) has biproducts for any pair of objects X and Y of $\mathcal{C}$.

Definition: If $\mathcal{A}$ and $\mathcal{B}$ are categories, then a *functor* F from $\mathcal{A}$ to $\mathcal{B}$ is a function that assigns to each $\mathcal{A}$-object A a $\mathcal{B}$-object F(A), and to each $\mathcal{A}$-morphism $f \in Hom(A, A')$ a $\mathcal{B}$-morphism $F(f) \in Hom(F(A), F(A'))$, in such a way that

1) F preserves composition; i.e., F(fog) = F(f) o F(g) whenever fog is defined,
2) F preserves identity morphisms; i.e., $F(id_A) = id_{F(A)}$ for each $\mathcal{A}$-object A.

Definition: A functor from a category to itself is called an *endofunctor* .i.e. its domain and codomain are the same.

Definition: An *additive functor* between pre-additive categories $\mathcal{A}$ and $\mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ such that for every two objects X and Y in $\mathcal{A}$, the function $Hom_{\mathcal{A}}(X,Y) \to Hom_{\mathcal{B}}(F(X); F(Y))$ is a homomorphism of abelian groups, i.e., F(f + g) = F(f) + F(g) for any morphisms $f, g : X \to Y$.

Definition: Let $\mathcal{C}$ be an additive category and $\Sigma : \mathcal{C} \to \mathcal{C}$ be an additive endofunctor of $\mathcal{C}$. Assume throughout that the endofunctor $\sum$ is invertible. A *candidate triangle* in $\mathcal{C}$ (with respect to $\sum$) is a diagram of the form:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

such that the composites vou, wov and $\sum uow$ are the zero morphisms.

Definition: A *morphism of candidate triangles* is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

where each row is a candidate triangle.

Pretriangulated Category

Definition: A *pretriangulated category* $\mathcal{T}$ is an additive category, together with an additive automorphism $\sum$, and a class of candidate triangles (with respect to $\sum$) called *distinguished triangles*. The following conditions must hold:

TR0:

a) Any candidate triangle which is isomorphic to a distinguished triangle is a distinguished triangle. (the distinguished triangles are closed under isomorphisms).
b) For any object X The candidate triangle

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

is distinguished.

TR1: For any morphism $f : X \to Y$ in $\mathcal{T}$ there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

TR2: Consider the two candidate triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

And

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y.$$

If one is a distinguished triangle, then so is the other.

TR3: For any commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

where the rows are distinguished triangles, there is a morphism $h : Z \to Z'$, **<u>not necessarily unique</u>**, which makes the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

commutative.

Remark .1 Let $\mathcal{T}$ be a pretriangulated category. Triangles in $\mathcal{T}$ are *stable* under isomorphism at any of their vertices, in the sense that if you replace one of X, Y, Z with an isomorphic object (and modify the morphisms appropriately) the result is still a triangle.

Remark.2 If $\mathcal{T}$ is a pretriangulated category then so is $\mathcal{T}^{\mathrm{op}}$ where we replace Σ by Σ^{-1}. We define the distinguished triangles of $\mathcal{T}^{\mathrm{op}}$ as follows: given a distinguished triangle of $\mathcal{T}$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

we define the following candidate triangle of $\mathcal{T}^{\mathrm{op}}$ (with respect to Σ-1) to be distinguished

$$\Sigma^{-1}Z \xleftarrow{\Sigma^{-1}w} X \xleftarrow{u} Y \xleftarrow{v} Z$$

With these structures, it is easy to check that $\mathcal{T}^{\mathrm{op}}$ is a pretriangulated category. Moreover the double dual $(\mathcal{T}^{\mathrm{op}})^{\mathrm{op}}$ is equal as a pretriangulated category to the original $\mathcal{T}$.

Definition: Let $\mathcal{T}$ be a pretriangulated category. Suppose that we are given a morphism of candidate triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}$$

There is a way to form a new candidate triangle out of this data. It is the Diagram

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma(Y \oplus X')$$

This new candidate triangle is called the *mapping cone* on a map of candidate triangles.

Definition: Let $\mathcal{T}$ be a pretriangulated category. Then $\mathcal{T}$ is *triangulated* if it satisfies the further hypothesis

TR4': Given any diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

where the rows are triangles, there is, by [TR3], a way to choose an $h : Z \to Z'$ to make the diagram commutative. This h may be chosen so that the mapping cone

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma(Y \oplus X')$$

is a triangle.

Example: **Vect**$_K$ **:**

Ob (Vect$_K$ **)** = all vector spaces over a fixed field K

Mor(Vect$_K$ **)**= K-linear transformations

The functor $\sum = \textit{id}$: **Vect**$_K$ **Vect**$_K$

distinguished triangles:

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X$$

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